QUANTIZATION OF GEOMETRY ASSOCIATED TO THE QUANTIZED KNIZHNIK-ZAMOLODCHIKOV EQUATIONS *

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Abstract.

It is known that solutions of the Knizhnik-Zamolodchikov differential equations are given by integrals of closed differential forms over suitable cycles. In this paper a quantization of this geometric construction is described leading to solution of the quantized Knizhnik-Zamolodchikov difference equations.

1. Introduction

The Knizhnik-Zamolodchikov equations (KZ) are fundamental differential equations discovered in conformal field theory by Knizhnik and Zamolodchikov in the beginning of the 80-s. It is known now that the KZ equations are the same equations as the differential equations for multidimensional hypergeometric functions. Hypergeometric functions are purely geometric objects and there is a certain geometry associated to these functions.

In mathematical physics the KZ differential equations were quantized. The quantized Knizhnik-Zamolodchikov equations (qKZ) are important difference equations [FR], [JM], [S]. It turns out that there is a geometric theory of the qKZ equations which is a quatization of the geometric theory of hypergeometric functions. This quantization of geometry is the subject of my talk.

First I will sketch the geometry of hypergeometric functions and then its quantization.

In this lecture I will tell about my joint work with V.Tarasov [TV2].

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2 KZ equations

Let $\mathfrak{g}$ be a simple Lie algebra, for instance sl_2. Let $V_1, \dots, V_n$ be $\mathfrak{g}$-modules, $V = V_1 \otimes \ldots \otimes V_n$. Let $\Omega \in \mathfrak{g} \otimes \mathfrak{g}$ be the tensor corresponding to an invariant scalar product. For $i < j$, let $\Omega_{i,j} : V \to V$ be the linear operator acting as Ω in $V_i \otimes V_j$ and as the

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identity operator in all of the other factors. The KZ equations on an V-valued function $\Psi(z_1, \dots, z_n)$ have the form

$$\frac{\partial \Psi}{\partial z_i} = \frac{1}{\kappa} \sum_{j, j \neq i} \frac{\Omega_{i,j}}{z_i - z_j} \Psi, \qquad i = 1, \dots, n.$$

Here κ is a parameter of the equation.

The KZ differential equations have very rich mathematical structures, they are closely connected to affine Lie algebras, quantum groups, topology of knots and three-folds.

3. Hypergeometric functions

There is a geometric source of differential equations of this type. These are differential equations for multidimensional hypergeometric functions.

Claim [SV, V].

The differential equations arising as differential equations for hypergeometric functions are "precisely" the same as the KZ equations.

Example of differential equations arising as differential equations for hypergeometric functions.

Fix numbers $a_1, \dots, a_n, \kappa$. Let

$$\Phi(t, z_1, \dots, z_n) = \prod_{\ell=1}^{n} (t - z_\ell)^{a_\ell / \kappa}.$$

For fixed $z_1, \dots, z_n$, consider the complex line $\mathbb{C}$ and denote by γ_m the oriented interval in $\mathbb{C}$ from z_m to z_{m+1}, $m = 1, \dots, n-1$.

Let

$$\Psi(z) = \Psi_\gamma(z_1, \dots, z_n) = (\int_\gamma \Phi \frac{dt}{t - z_1}, \dots, \int_\gamma \Phi \frac{dt}{t - z_n})$$

where $\gamma = \gamma_1, \dots, \gamma_{n-1}$. Then

$$\frac{\partial \Psi}{\partial z_i} = \frac{1}{\kappa} \sum_{j, j \neq i} \frac{\Omega_{i,j}}{z_i - z_j} \Psi, \qquad i = 1, \dots, n,$$

where $\Omega_{i,j}(a)$ are some matrices independent on z and γ. Different γ give different solutions to the same equations.

Geometry of hypergeometric functions

For fixed $z_1, \dots, z_n$, consider a complex

$$0 \to \Omega^0 \to \Omega^1 \to 0,$$

where

$$\Omega^0 = \{ f(t)\Phi(t,z) \,|\, f \text{ is any rational function regular on } \mathbb{C} - \{z_1, \dots, z_n\} \},$$

$$\Omega^1 = \{ f(t)\Phi(t,z)dt \,|\, f \text{ is any rational function regular on } \mathbb{C} - \{z_1, \dots, z_n\} \},$$

and the differential of the complex is the standard differential.

Theorem.
For generic $a_1, \dots, a_n, \kappa$, *we have* $H^0 = 0$, *dim* $H^1 = n-1$, *and the differential forms* $\Phi dt/(t-z_1), \dots, \Phi dt/(t-z_{n-1})$ *form a basis in* H^1.

Let $H_1 = (H^1)^*$ be the dual space, the first homology group. Each interval γ_m, $m = 1, \dots, n-1$, defines a linear function on Ω^1,

$$[\gamma_m] \,:\, f\Phi dt \mapsto \int_{\gamma_m} f\Phi dt.$$

By the Stokes' theorem we have $\int_{\gamma_m} d(f\Phi) = 0$ and , hence, $[\gamma_m]$ defines an element of H_1.

Theorem.
The elements $[\gamma_m], m = 1, \dots .n-1$, *form a basis in* H_1.

This fact is based on the formula

$$\det{}_{1\le \ell, m \le n-1}\Big(\frac{a_\ell}{\kappa}\int_{\gamma_m} \Phi\,\frac{dt}{t-z_\ell}\Big) = \frac{\Gamma(\frac{a_1}{\kappa}+1)\dots\Gamma(\frac{a_n}{\kappa}+1)}{\Gamma(\frac{a_1+\dots+a_n}{\kappa}+1)}\prod_{i\ne j}(z_i-z_j)^{a_j/\kappa}.$$

For $n = 2$, $(z_1, z_2) = (0, 1)$, the formula takes the form

$$b\int_0^1 t^a(1-t)^{b-1}dt = \frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+1)}.$$

We have a correspondence: conformal field theory corresponds to geometry of hypergeometric functions, the space of conformal blocks corresponds to the cohomology space H^1 associated to the function $\Phi(t, z)$, the KZ equations correspond to the differential equations for integrals of basic closed differential forms (to the Gauss-Manin connection), solutions of the KZ equations correspond to cycles $\gamma \in H_1$.

Remark. More general solutions of the KZ equations are constructed geometrically in a similar way starting from a more general function

$$\Phi(t_1, \dots, t_k, z_1, \dots, z_n) \;=\; \prod(t_i - z_m)^{a_{i,m}/\kappa}\prod(t_i-t_j)^{b_{i,j}/\kappa}.$$

5. qKZ equations

The KZ differential equations were quantized in [S], [FR], [JM]. The qKZ equations are difference equations satisfied by form factors in [S], by matrix elements of intertwining operators in [FR], by correlation functions of statistical models in [JM].

There are three versions of the qKZ equations: rational, trigonometric and elliptic. We discuss here the rational version. Let $V_1, \dots, V_n$ be sl_2 modules with highest weights, $V = V_1 \otimes \dots \otimes V_n$. The rational qKZ on an V-valued function $\Theta(z_1, \dots, z_n)$ has the form

$$\Theta(z_1, \dots, z_m + p, \dots, z_n) = K_m(z_1, \dots, z_n, p)\,\Theta(z_1, \dots, z_n), \qquad m = 1, \dots, n,$$

where p is the step of the equations, $K_m : V \to V$ are linear operators defined as follows.

For any i, j there is a a matrix called the rational R-matrix

$$R_{V_i,V_j}(x) \,:\, V_i \otimes V_j \,\to\, V_i \otimes V_j$$

sutisfying the Yang-Baxter equation

$$R_{V_i,V_j}(x) R_{V_i,V_k}(x+y) R_{V_j,V_k}(y) = R_{V_j,V_k}(y) R_{V_i,V_k}(x+y) R_{V_i,V_j}(x)$$

and normalized by the condition

$$R_{V_i,V_j}(x) \,:\, v_i \otimes v_j \,\to\, v_i \otimes v_j \,,$$

where v_i, v_j are highest weight vectors, see [JKMO],[KRS]. Then

$$K_m = R_{V_m,V_{m-1}}(z_m - z_{m-1} + p) \dots R_{V_m,V_1}(z_m - z_1 + p)\, R_{V_m,V_n}(z_m - z_n) \dots R_{V_m,V_{m+1}}(z_m - z_{m+1}).$$

The rational qKZ difference equation turns into the KZ differential equation under the following limiting procedure.

Let $z_m = SZ_m$, where $S \gg 1$ is a number, Z_m are new variables. Then for the new variables and a new function $\Psi(Z) := \Theta(Sz)$ we have $\Psi(\dots, Z_m + p/S, \dots) = K_m(SZ, p)\, \Psi(Z_1, \dots, Z_n)$ and

$$K_m(SZ, p) = 1 + \frac{1}{S} \sum_{j \neq m} \frac{\Omega_{m,j} + c_{m,j}}{Z_m - Z_j} + \mathcal{O}(\frac{1}{S^2}),$$

where $c_{m,j}$ are suitable numbers. Hence, the difference equation turns into the differential equation

$$\frac{\partial \Psi}{\partial Z_m} = \frac{1}{p} \sum_{j \neq m} \frac{\Omega_{m,j} + c_{m,j}}{Z_m - Z_j}\, \Psi$$

when S tends to infinity.

6. Solutions to the QKZ equations and eigenvectors of commuting Hamiltonians

Assume that p tends to zero and

$$\Theta(z, p) = e^{S(z)/p}(f(z) + p f_1(z) + \dots).$$

Then

$$K_m(z, p = 0) f(z) = \lambda_m(z) f(z),$$

where $\Lambda_m(z) = e^{\partial S(z)/\partial z_\ell}$. This means that the leading term $f(z)$ is an eigenvector of the commuting operators $K_m(z, p = 0), m = 1, \dots, n$. The operators K_m are Hamiltonians of a quantum spin chain model . Having a solution to the qKZ equations and computing its quasiclassical asymptotics one can construct eigenvectors of Hamiltonians. These eigenvectors coincide with the Bethe vectors constructed by the Bethe ansatz [TV1].

7. Solutions to the qKZ equations

As we know the KZ differential equations are realized as differential equations for closed differential forms over cycles depending on parameters. It turns out that there is a quantization of all these geometric objects leading to solutions of the qKZ difference equations.

Let us describe a p-analog of the function $\Phi(t,z) = \prod(t-z_\ell)^{a_\ell/\kappa}$

The function $T^{a/\kappa}$ satisfies the differential equation

$$\frac{dY}{dT} = \frac{a}{\kappa} Y. \tag{1}$$

A p-analog of this differential equation is the difference equation

$$y(t+p) = \frac{t+a}{t-a}\, y(t). \tag{2}$$

If $t = ST$, $Y(T) := y(ST)$, then the difference equation takes the form

$$Y(T+\frac{p}{S}) = (1 + \frac{2a}{S}\frac{1}{T-a/S})\, Y(T)$$

and turns into the differential equation (1) with $\kappa = p/2$ as S tends to infinity.

Equation (2) has a solution

$$y = \Gamma(\frac{t+a}{p})\Gamma(1-\frac{t-a}{p})e^{-\pi i t/p}.$$

Introduce a function

$$\Phi_p(t,z) = \prod_{\ell=1}^{n}\Gamma(\frac{t-z_\ell+a_\ell}{p})\Gamma(1-\frac{t-z_\ell-a_\ell}{p})e^{-\pi i(t-z_\ell)/p}.$$

This function is a p-analog of the function Φ of section 3.

Properties.

1.
$$\Phi_p(t+p,z) = \prod_{\ell=1}^{n}\frac{t-z_\ell+a_\ell}{t-z_\ell-a_\ell}\,\Phi_p(t,z) =: b_0(t,z)\Phi_p(t,z).$$

2.
$$\Phi_p(t,z_1,\dots.,z_\ell+p,\dots,z_n) = \frac{t-z_\ell-a_\ell-p}{t-z_\ell+a_\ell-p}\,\Phi_p(t,z) =: b_\ell(t,z)\Phi_p(t,z).$$

We have

$$b_m(\dots,z_\ell+p,\dots)\,b_\ell(z) = b_\ell(\dots,z_m+p,\dots)\,b_m(z),$$

where $\ell, m \in \{0,\dots,n\}$.

The singularities of $\Phi(t,z)$ for fixed z are the points $\{t\in\mathbb{C}\,|\,t = z_\ell - a_\ell - Np$, $t = z_\ell + a_\ell + (N+1)p,\ \text{where}\, \ell = 1,\dots,n,\ N = 0,1,2,\dots\}$.

Denote by $Sing^\vee(z)$ the set $\{t\in\mathbb{C}\,|\,t\in\mathbb{C}\,|\,t = z_\ell - a_\ell + (N+1)p$, $t = z_\ell + a_\ell - Np,\ \text{where}\, \ell = 1,\dots,n,\ \ell = 0,1,2,\dots\}$.

Denote by $F(z)$ the linear space of rational functions $f:\mathbb{C}\to\mathbb{C}$, regular on $\mathbb{C} - Sing^\vee(z)$ and with at most simple poles at $Sing^\vee(z)$.

Consider a complex

$$0\ \to F(z)\ \to\ F(z)\ \to 0,\ f(t)\ \mapsto\ f(t+p)b_0(t,z) - f(t).$$

Property. The differential of this complex, $D_p(z)$, preserves the space $F(z)$.

Introduce the p-cohomology group by $H^1_{(p)}(z) = F(z)/D_p(z)F(z)$.

Introduce linear maps

$$B_\ell(z) : F(\dots, z_\ell + p, \dots) \to F(\dots, z_\ell, \dots),\ f(t) \mapsto f(t)b_\ell(t,z)$$

where $\ell = 1, \dots, n$.

Properties.

1. $B_\ell(z)$ is an isomorphism.
2. $B_\ell(z)$ commutes with the differential D_p.
3. $B_m(\dots, z_\ell + p, \dots)B_\ell(z) = B_\ell(\dots, z_m + p, \dots)B_m(z)$.

Corollary. *There are well defined isomorphisms*

$$B_\ell(z) : H^1_{(p)}(\dots, z_\ell + p, \dots) \to H^1_{(p)}(\dots, z_\ell, \dots).$$

These isomorphisms satisfy property 3.

Corollary. *If $z - z' \in (p\mathbb{Z})^n$, then $H^1_{(p)}(z)$ and $H^1_{(p)}(z')$ are canonically identified.*

We call these isomorphisms a *discrete flat connection.*

Let $H_1^{(p)}(z)$ be the space dual to $H^1_{(p)}(z)$. The space will be called the *p-homology group*, its elements will be called *p-cycles.*

Let $\gamma : z \mapsto \gamma(z) \in H_1^{(p)}(z)$ be a section. The section is called *periodic* if $B_\ell(z)\,\gamma(\dots, z_\ell + p, \dots) = \gamma(z)$. We define periodic sections with values in $H^1_{(p)}(z)$ analogously .

Theorem.

For generic $z_1, \dots, z_n, a_1, \dots, a_n, p$, the dimension of $H_1^{(p)}(z)$ equals $n-1$. The elements

$$w_j(z,t) = \frac{1}{t - z_j - a_j} \prod_{\ell=1}^{j-1} \frac{t - z_\ell + a_\ell}{t - z_\ell - a_\ell} \ \in\ F(z), \qquad j = 1, \dots, n-1,$$

generate a basis in $H^1_{(p)}(z)$. Under the above limit $S \to \infty$, the element w_j tend to $1/(T - Z_j)$.

8. Difference equations of the discrete connection

Since the elements $[w_1(z)], \dots, [w_{n-1}(z)]$ form a basis in $H_1^{(p)}(z)$, the isomorphisms $B_\ell(z)$ could be given by matrices $\beta_\ell(z)$,

$$B_\ell(z)[w_j(\dots, z_\ell + p, \dots)] = w_i(z)\,\beta^i_{\ell,j}(z).$$

For any p-cycle $\gamma \in H^1_{(p)}(z)$ introduce its coordinate vector

$$I = (\langle w_1(z), \gamma\rangle, \dots, \langle w_{n-1}(z), \gamma\rangle).$$

Theorem.

Let a p-cycle $\gamma(z) \in H_1^{(p)}(z)$ be periodic, then its coordinate vector satisfies the system of difference equations

$$I(\dots, z_\ell + p, \dots) = I(z)\beta_\ell(z), \qquad , \ell = 1, \dots, n. \tag{3}$$

Changing periodic cycle γ, one constructs all solutions to system (3).

Proof. $I(\dots, z_\ell + p, \dots) = (\dots, \langle w_j(\dots, z_\ell + p, \dots), \gamma(\dots, z_\ell + p, \dots)\rangle, \dots) =$
$(\dots, \langle B_\ell(z) w_j(\dots, z_\ell + p, \dots), (B_\ell(z)^*)^{-1}\gamma(\dots, z_\ell + p, \dots)\rangle, \dots) =$
$(\dots, \langle w_i(z)\, \beta^i_{\ell,j}, \gamma(z)\rangle, \dots) = I(z)\beta_\ell(z)$.

Theorem. *System (3) coincides with a special case of the qKZ equations. This special case is the quantization of the special case of the KZ differential equations associated to the function $\Phi(t,z) = \prod (t - z_\ell)^{a_\ell/\kappa}$ of section 3.*

All qKZ equations associated to representations of sl_n could be constructed in a similar way.

Remark. The theorem says that the p-homology classes can be naturally identified with suitable vectors in the corresponding tensor product of representations of sl_2 in such a way that the action of R-matrices is identified with the action of the discrete connection.

9. p-Homology theory

According to our definitions

$$D_p(z) : f(t) \mapsto (f(t+p)\Phi_p(t+p, z) - f(t)\Phi_p(t,z))/\Phi_p(t,z),$$

$H^1_{(p)}(z) = F(z)/D_p(z)F(z)$ and $H_1^{(p)}(z) = (H^1_{(p)}(z))^*$. In this section we will construct linear functions on $H^1_{(p)}(z) = F(z)/D_p(z)F(z)$, that is, we will construct elements of $H_1^{(p)}(z)$.

A naive idea.

For $\xi \in \mathbb{C}$ and a function $h : \mathbb{C} \to \mathbb{C}$ with compact support define the Jackson integral by

$$\int_{[\xi]_p} h\, d_p t = p \sum_{\ell=-\infty}^{\infty} h(\xi + \ell p).$$

Property. Let h be of the form $h(t) = D_p g(t) = g(t+p) - g(t)$, then

$$\int_{[\xi]_p} h\, d_p t = p \sum_{\ell=-\infty}^{\infty} (g(\xi + (\ell+1)p) - g(\xi + \ell p)) = 0.$$

Therefore, a naive way to construct elements of $H_1^{(p)}(z)$ would be to take any ξ and define for any $f \in F(z)$

$$< f, [\xi]_p >= \int_{[\xi]_p} \Phi_p(t,z) f(t) d_p t.$$

Then such a linear function on $F(z)$ would be zero on $D_p(z)F(z)$ and would define an element of $H_1^{(p)}(z)$.

Unfortunately, this idea does not work since the integral does not converge. However, a certain modification of the idea could be realized.

Choose p so that p is imaginary, $p \in i\mathbb{R}$, and $\mathrm{Im}\, p > 0$. Choose $z_1, \dots, z_n$ to be real, and choose $a_1, \dots, a_n$ to be imaginary and lying in the upper half plane, $a_\ell \in i\mathbb{R}$, and $\mathrm{Im}\, a_\ell > 0$.

Under these assumptions the real line $\mathbb{R}$ separates in $\mathbb{C}$ the poles of the factors $\prod_{\ell=1}^n \Gamma(\frac{t-z_\ell+a_\ell}{p})$ and the factors $\prod_{\ell=1}^n \Gamma(1-\frac{t-z_\ell-a_\ell}{p})$ of the function Φ_p.

Let $G_m : \mathbb{C} \to \mathbb{C}$, $t \mapsto e^{2\pi i m t/p}$, $m = 1, \dots, n-1$. The functions G_m are the simplest p-periodic functions.

For every such m, consider a map

$$[G_m] : F(z) \to \mathbb{C}, \;\; f \mapsto \int_{\mathbb{R}} G_m(t)\Phi_p(t,z)f(t)dt.$$

As we will see, the maps $[G_m]$ are p-analogs of the intervals γ_m.

Properties of the linear functionals $[G_m]$**.**

1. The functionals $[G_m]$, $m = 1, \dots, n-1$, are well defined. Moreover, for any other integer m, the linear functional $[G_m]$ is not defined on $F(z)$.

Proof. The function Φ_p is a product of gamma functions. When $x \to \infty$, the gamma function $\Gamma(x)$ has asymptotics

$$\Gamma(x) = x^{-1/2} e^{x(\ln x \,-\, 1)} (2\pi)^{1/2} + \dots$$

These asymptotics imply the property.

2. For $m = 1, \dots, n-1$, $[G_m]|_{D_p(z)F(z)} = 0$.

Proof.

$$\int_{\mathbb{R}} G_m(t)[\Phi_p(t+p,z)f(t+p) - \Phi_p(t,z)f(t)]\, dt =$$
$$\int_{\mathbb{R}+p} G_m(t)\Phi_p(t,z)f(t)\, dt - \int_{\mathbb{R}} G_m(t)\Phi_p(t,z)f(t)\, dt = 0.$$

Here we use periodicity of the function $G_m(t)$ and the fact that there are no poles of the integrand between $\mathbb{R}$ and $\mathbb{R}+p$.

Corollary. The functional $[G_m]$ defines an element of $H_1^{(p)}(z)$.

3. If $z_\ell = SZ_\ell$ and $S \to \infty$, then the linear functional $[G_m]$ tends to the interval $[Z_m, Z_{m+1}]$ in the following sence:

$$\int_{\mathbb{R}} G_m(t)\Phi_p(t,z)w_\ell(t)\, dt = C_m(S,p)\, (\int_{Z_m}^{Z_{m+1}} \prod_{j=1}^n (T-Z_j)^{2a_j/p} \frac{dT}{T-Z_\ell} \; + O(1/S))$$

for every ℓ. Here C_m is an explicitly given function.

Proof follows from the Stirling formula.

4. The sections $[G_m](z) \in H_1^{(p)}(z)$ are p-periodic.

Proof. For $f \in F(\dots, z_\ell + p, \dots)$ we have $\langle [G_m](\dots, z_\ell+p, \dots), [f]\rangle :=$ $\int_{\mathbb{R}} G_m(t)\, \Phi_p(t, \dots, z_\ell + p, \dots)\, f(t)\, dt = \int_{\mathbb{R}} G_m(t)\, \Phi_p(t,z)\, b_\ell(t,z)\, f(t)\, dt =$ $\langle [G_m](z), B_\ell(z)[f]\rangle$.

5. The elements $[G_m](z)$, $m = 1, \dots, n-1$, form a basis in $H_1^{(p)}(z)$.

Corollary. The vectors

$$\Theta_m(z) = (\int_{\mathbb{R}} G_m \Phi_p w_1 dt, \ldots, \int_{\mathbb{R}} G_m \Phi_p w_{n-1} dt), \qquad m = 1, \ldots, n-1,$$

form a basis of solutions to the difference equations of the discrete connection on $H^1_{(p)}(z)$, that is a basis of solutions to the corresponding qKZ equations.

Property 5 follows from the following property 6.

6.

$$\det_{1\leq m,\ell\leq n-1} (\frac{2a_\ell}{p} \int_{\mathbb{R}} G_m \Phi_p w_\ell dt) = e^{(n-1)\pi i \sum_{j=1}^n z_j/p} (2\pi i)^{n(n-1)/2} \times$$

$$\times \frac{\prod_{j=1}^n \Gamma(\frac{2a_j}{p}+1)}{\Gamma(\frac{2}{p}\sum_{j=1}^n a_j \ +1)} \prod_{\ell<m} \Gamma(\frac{z_m + a_m - z_\ell + a_\ell}{p}) \Gamma(1 - \frac{z_\ell + a_\ell - z_m + a_m}{p}).$$

Example. For $n=2$, we have Barnes' formula [WW]

$$\int_{-i\infty}^{i\infty} \Gamma(a+t)\Gamma(b+t)\Gamma(c-t)\Gamma(d-t) dt = 2\pi i \frac{\Gamma(a+c)\Gamma(a+d)\Gamma(b+c)\Gamma(b+d)}{\Gamma(a+b+c+d)}.$$

10. Conclusion

The theory of differential equations is a beautiful and well developed theory. The theory of difference equations is less elaborate. One of the reasons is absence of well posed problems, good examples, absence of indications to how to extend the notions of the theory of differential equations to the theory of difference equations. Mathematical physics (field theory, statistical mechanics) indicates such examples and a passage from differential to difference equations. In particular, mathematical physics indicates a surprising quantization of geometry in which algebraic functions are replaced by gamma functions, such geometric objects as cycles are replaced by exponential functions and all basic relations remain preserved.

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